

SECRETARY PROBLEM WITH QUALITY-BASED PAYOFF

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ABSTRACT. We consider a variant of the classical Secretary Problem. In this setting, the candidates are ranked according to some exchangeable random variable and the quest is to maximize the expected quality of the chosen aspirant. We find an upper bound for the optimal hiring rule, present examples showing it is sharp, and recover the classical case, among other results.

INTRODUCTION

A recruiter is faced with the task of selecting the best assistant among a stream of n applicants, on a reject-or-hire basis. Namely, the decision is made right after the interview, there is no coming back once a candidate is rejected, and all the information gathered during the interview is whether the current postulant is or not better than all its precursors. Since there is no *a priori* information about the applicants, the best strategy for the interviewer implies establishing a *threshold* and selecting the first candidate that arrives after such point and is the best the recruiter has interviewed so far. In the classical Secretary Problem the question is to find the optimal threshold, provided that we want to maximize the probability of hiring the best aspirant.

In this nice introductory example of a statistical decision making problem one learns that the best strategy is to blindly reject the first $\approx n/e$ candidates and from that point on, to select the first postulant that is superior than all of the previous ones. If none is chosen with this plan, one just hires the last applicant. We consider n , the total number of candidates, as a known quantity, that all of them are totally ranked with no ties, the recruiter is only allowed to determine if the current aspirant is the best that has arrived so far, and that the order in which applicants arrive is random.

For a historical overview of this problem, some generalizations, and a conjecture about Kepler's choice of his second wife, see the both interesting and fun article by Ferguson [4].

Among its most known variants there is the Post-doc Problem, under the assumption that success is achieved when the selected applicant is the second best, (considering that the best one will go to Harvard anyway); the Problem of admitting a Class of Students, instead of only one, from an aspirant pool, in which the task is to find a subset of candidates all of them better ranked than all the rejected ones in an on-line algorithm (see [7]); and the Problem of selecting the best k secretaries out of n with a similar method [6]. The Odds-theorem [2] provides another

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framework in which the classical Secretary Problem may be solved and also allows to handle with group interviews.

In the present article we tackle a variant of the Secretary Problem in which the goal is to maximize the expected value of the quality of the selected applicant. Bearden [1] proves that the optimal threshold for independent and identically distributed (i.i.d.) *uniform* random variables is the integer closest to \sqrt{n} .

As a by-product of a fruitful discussion after a talk about the work of Bearden in the inconspicuous 2038–seminar, we consider different distributions and prove that, in general, the optimal threshold $c_*(n)$ is essentially bounded from above by n/e (see Theorem 2.1 below for details). We provide several examples for both continuous and discrete distributions, and study the behavior of the optimal threshold in each situation. Among these examples, we recover the classical Secretary Problem (Example 4.5) and show that the bound from Theorem 2.1 is sharp (see examples 4.3, 4.5 and 4.6). We also prove the complete monotonicity of the order statistics along the way. The aim of these notes is to give a detailed account of what has been achieved in the aftermath of the aforementioned seminar.

The paper is organized as follows. In Section 1 we give the general setting of the problem, and define the optimal threshold as well (Definition 1.1 therein), in Section 2 we state and prove the main result, namely the upper bound of the optimal stopping rule regardless of the distribution chosen (Theorem 2.1). Afterwards, in Section 3, we study the behavior of $c_*(n)$ as $n \rightarrow \infty$, and finally in Section 4 we provide several examples as Exponential, Normal, Pareto distributions, together with permutations and Bernoulli variables, and explain how to recover the classical problem from our framework.

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1. PRELIMINARIES AND NOTATION

Consider the following variant of the secretary problem. A recruiter wants to hire an assistant among n candidates. Suppose that the qualifications of the applicants are given by exchangeable random variables $(X_k)_{k=1,\dots,n}$ with finite expected value and that the recruiter is aware of their joint distribution. The candidates start arriving to the interview, one by one, and the only information the recruiter is able to gather is whether the current applicant is the best one evaluated so far. If $\mathbb{P}\{X_i = X_j\} = 0$ for $i \neq j$, there are no ambiguities when deciding whether a candidate is unrivaled hitherto. Otherwise, we have to define properly what happens if the current applicant is as good as the most qualified interviewed by then. To this end, we rank the prospects by quality, break ties at random and regard a candidate as unsurpassed thus far if he or she is the best one according to this rank. Once the interview is finished, the recruiter has to choose whether rejecting or hiring the applicant. The objective of the recruiter is to maximize the expected value of the quality of the hired assistant. As in the classical Secretary Problem, the strategies that make sense follow the same schemes: the recruiter determines a threshold $c \in \{1, \dots, n\}$, then rejects the first $c - 1$ candidates, and from that point on decides to hire the prospect if, when it comes to the interview,

it is the topmost as yet. In case the recruiter does not find any applicant satisfying these conditions he simply hires the last one.

With this strategy, the expected value for $c \geq 2$ is given by

$$(1.1) \quad V_n(c) = \sum_{k=c}^{n-1} \mathbb{E}[X_k | X_k = \max\{X_1, \dots, X_k\}] \frac{c-1}{k-1} \frac{1}{k} + \frac{c-1}{n-1} \mathbb{E}[X_n].$$

Indeed, we are summing the expected value for the k -th applicant (provided it surpasses all the former ones) times the probability of being selected, and including at the end of this formula the case in which the recruiter hires the last one. Observe that the k -th candidate is selected if and only if the best one among the first $k-1$ is in the group of the first $c-1$ applicants and the k -th is superior to its precursors. Moreover, we set $V_n(1) := \mathbb{E}[X_1]$.

Let us define

$$\mu_k := \mathbb{E}[X_k | X_k = \max\{X_1, \dots, X_k\}]$$

and

$$\mu := \mathbb{E}[X_k].$$

These values are well defined and are finite since the random variables are assumed to be identically distributed and to have finite expected value. With this notation, we may rewrite (1.1) as

$$V_n(c) = \sum_{k=c}^{n-1} \mu_k \frac{c-1}{k-1} \frac{1}{k} + \frac{c-1}{n-1} \mu.$$

The discrete derivative of V_n , defined by the forward difference, has the following expression

$$(1.2) \quad \Delta V_n(c) := V_n(c+1) - V_n(c) = \sum_{k=c+1}^{n-1} \frac{\mu_k}{k(k-1)} - \frac{\mu_c}{c} + \frac{\mu}{n-1}$$

valid for $c \geq 1$. We aim to find an expression for $V_n(c+1) - V_n(c)$ suitable for algebraic manipulations. In order to do so, let us recall the summation by parts formula

$$\sum_{k=m}^n f_k(g_{k+1} - g_k) = [f_{n+1}g_{n+1} - f_m g_m] - \sum_{k=m}^n g_{k+1}(f_{k+1} - f_k).$$

This enables us to rewrite (1.2) as

$$(1.3) \quad \Delta V_n(c) = \sum_{k=c}^{n-1} \frac{\mu_{k+1} - \mu_k}{k} - \frac{\mu_n - \mu_1}{n-1}.$$

Therefore, the discrete second derivative of V_n becomes

$$\Delta^2 V_n(c) := V_n(c+2) - 2V_n(c+1) + V_n(c) = -\frac{\mu_{c+1} - \mu_c}{c} \leq 0.$$

The last inequality holds because $\mu_k \leq \mu_{k+1}$ for every k , since the exchangeability of the random variables implies $\mu_k = \mathbb{E}[\max\{X_1, \dots, X_k\}]$. This proves that V_n is concave in c and therefore, local maxima are automatically global maxima. In conclusion, the desired c is any one that satisfies $\Delta V_n(c) \leq 0$ and $\Delta V_n(c-1) \geq 0$. For any such c , the strategy of rejecting the first $c-1$ candidates and, after that,

hiring an applicant if it is the best so far, maximizes the expected value of the candidate hired.

Definition 1.1. *The optimal threshold for n secretaries is*

$$c_*(n) := \min\{c \geq 1 : \Delta V_n(c) \leq 0\}.$$

Remark 1.2. It could be the case that there exists more than one c maximizing $V_n(c)$; if this occurs, we take the minimum among all such values. Consequently, whenever we set an upper bound for the threshold we are stating that there exists a c in the set of maximizers that satisfies it, and whenever we provide a lower bound, it means that it is satisfied for every maximizer. Moreover, setting c_* as the lowest possible is natural when considering the problem from the point of view of the recruiter, who is seeking to maximize the quality of the selected candidate and not to perform too many interviews.

2. MAIN THEOREM

After having defined the optimal threshold, the reader may ask himself about the dependence of $c_*(n)$ on the distribution of the random variables $(X_k)_{k=1,\dots,n}$. It is well known that for the classical Secretary Problem the optimal threshold is given by n/e . We claim that this number is an upper bound for $c_*(n)$, regardless of the distributions $(X_k)_{k=1,\dots,n}$ of the candidates. More precisely, in this Section we prove the following theorem.

Theorem 2.1. *Given any set of exchangeable random variables $(X_k)_{k=1,\dots,n}$, the optimal threshold $c_*(n)$ satisfies*

$$(2.1) \quad \sum_{k=c_*(n)-1}^{n-1} \frac{1}{k} > 1.$$

In particular, the bound $c_(n) \lesssim n/e$ holds.*

In order to prove the previous theorem, let us define

$$\mu_{k:n} := \mathbb{E}[X_k | X_1 \leq \dots \leq X_n].$$

Since the random variables under consideration are exchangeable, it is clear that $\mu_k = \mu_{k:k}$. Given the sequence $(\mu_k)_{k=1,\dots,n}$, consider the discrete derivatives

$$\Delta \mu_k = \Delta^1 \mu_k = \mu_{k+1} - \mu_k,$$

and in general

$$\Delta^j \mu_k = \Delta^{j-1} \mu_{k+1} - \Delta^{j-1} \mu_k.$$

Straightforward computations lead to the identity

$$(2.2) \quad \Delta^j \mu_k = \sum_{i=0}^j \mu_{k+i} \binom{j}{i} (-1)^{j-i} = \sum_{i=k}^{k+j} \mu_i \binom{j}{i-k} (-1)^{j-(i-k)}, \text{ if } k+j \leq n.$$

Now we prove the complete monotonicity of the order statistics.

Proposition 2.2. *Let $k, j \geq 0$ be such that $k+j \leq n$, then the following formula holds:*

$$\Delta^j \mu_k = \binom{k+j}{j}^{-1} (-1)^{j+1} (\mu_{k+1:k+j} - \mu_{k:k+j}).$$

Proof. If we consider exchangeable random variables $(X_l)_{l=1,\dots,k+l}$, then every relative order between them is equally likely. By considering the possible ranks of the top-ranked variable among the first i when taking into account the relative order of all the variables, we obtain

$$\mu_i = \binom{k+j}{i}^{-1} \sum_{l=i}^{k+j} \mu_{l:k+j} \binom{l-1}{i-1}.$$

Next, we replace this expression on the right hand side of (2.2). After interchanging the order of summation and some algebraic manipulation, we obtain

$$(2.3) \quad \Delta^j \mu_k = \frac{j!(-1)^j}{(k+j)!} \sum_{l=k}^{k+j} \mu_{l:k+j} (l-1)! \sum_{i=0}^{l-k} \frac{(k+i)(-1)^i}{(l-k-i)!i!}.$$

Let us recall that

$$\sum_{i=0}^t (-1)^i \binom{t}{i} = \delta_t \quad \text{and} \quad \sum_{i=0}^t i(-1)^i \binom{t}{i} = -\delta_{t-1}.$$

Therefore, the last summation of (2.3) becomes

$$\begin{aligned} \sum_{i=0}^{l-k} \frac{(k+i)(-1)^i}{(l-k-i)!i!} &= \frac{1}{(l-k)!} \left(k \sum_{i=0}^{l-k} (-1)^i \binom{l-k}{i} + \sum_{i=0}^{l-k} i(-1)^i \binom{l-k}{i} \right) \\ &= \frac{1}{(l-k)!} (k\delta_{l-k} - \delta_{l-k-1}). \end{aligned}$$

Plugging this last expression into (2.3) we get

$$\Delta^j \mu_k = \frac{j!(-1)^j}{(k+j)!} (\mu_{k:k+j} (k-1)!k - \mu_{k+1:k+j} k!)$$

from which the result follows. \square

An immediate consequence of the previous proposition is the following.

Corollary 2.3. *Let $j+k \leq n$. Then, $\Delta^j \mu_k \geq 0$ if j is odd and $\Delta^j \mu_k \leq 0$ if j is even.*

Moreover, setting $j=2$ in Corollary 2.3, we obtain:

Corollary 2.4. *The sequence $\mu_{k+1} - \mu_k$ is decreasing.*

At this point we are ready to provide a proof of our main result.

Proof of Theorem 2.1. Replacing $\mu_n - \mu_1$ by

$$\sum_{k=1}^{n-1} \mu_{k+1} - \mu_k$$

in (1.3) gives

$$\Delta V_n(c) = \sum_{k=c}^{n-1} (\mu_{k+1} - \mu_k) \left(\frac{1}{k} - \frac{1}{n-1} \right) + \sum_{k=1}^{c-1} (\mu_{k+1} - \mu_k) \left(\frac{-1}{n-1} \right),$$

whose right hand side is not greater than

$$\sum_{k=c}^{n-1} (\mu_{c+1} - \mu_c) \left(\frac{1}{k} - \frac{1}{n-1} \right) + \sum_{k=1}^{c-1} (\mu_{c+1} - \mu_c) \left(\frac{-1}{n-1} \right),$$

from which we conclude

$$\Delta V_n(c) \leq (\mu_{c+1} - \mu_c) \left(\sum_{k=c}^{n-1} \frac{1}{k} - 1 \right).$$

Then, $\Delta V_n(c) \leq 0$ when

$$\sum_{k=c}^{n-1} \frac{1}{k} \leq 1,$$

which proves that $c_*(n)$ satisfies property (2.1). \square

Remark 2.5. The upper bound given by (2.1) is sharp, as it is attained by examples 4.3, 4.5 and 4.6 below. On the other hand, there are no non-trivial lower bounds for the optimal threshold valid in general for any i.i.d. random variables. The idea behind this fact is that if almost every candidate has maximal quality, then the recruiter has no need to wait. We refer to Example 4.6 and Remark 4.7 for a simple construction in which the optimal threshold is $c_* = 2$ for any number of applicants.

3. ASYMPTOTIC RESULTS FOR INDEPENDENT VARIABLES

Throughout this section we make the further assumption that for each n the random variables $(X_k)_{1 \leq k \leq n}$ are independent with a given distribution. We disregard the case of a Dirac delta distribution in which all the candidates are equally suitable, and thus every strategy furnishes the same result.

In this setting it is interesting to study the behavior of $c_*(n)$ as n varies. We prove that c_* is a non-decreasing function of n that diverges as n goes to infinity. This means that the recruiter has to wait longer as n grows and that $c_*(n)$ becomes as large as wanted. Namely, given any $m \geq 1$, the interviewer would have to reject the first m candidates if n (the total number of applicants) is taken large enough.

Proposition 3.1. *The optimal threshold grows with the amount of candidates, that is, $c_*(n) \leq c_*(n+1)$.*

Proof. Given $c < c_*(n)$, using (1.3) we obtain

$$0 < \Delta V_n(c) = \sum_{k=c}^{n-1} \frac{\mu_{k+1} - \mu_k}{k} - \frac{\mu_n - \mu_1}{n-1}.$$

Employing again (1.3), now for c and $n+1$ we obtain

$$\begin{aligned} \Delta V_{n+1}(c) &= \Delta V_n(c) + \frac{\mu_{n+1} - \mu_n}{n} + \frac{\mu_n - \mu_1}{n-1} - \frac{\mu_{n+1} - \mu_1}{n} \\ &= \Delta V_n(c) + \frac{\mu_n - \mu_1}{n(n-1)} > 0. \end{aligned}$$

Then, $\Delta V_{n+1}(c) > 0$, so $c < c_*(n+1)$, implying that $c_*(n) \leq c_*(n+1)$ as we claimed. \square

Lemma 3.2. *The sequence $\{\mu_k\}_{k=1}^\infty$ is strictly increasing.*

Proof. If the Cumulative Distribution Function (CDF) of the variables is $F(x)$, then the CDF of the maximum of k is $F^k(x)$. Therefore,

$$\mu_{k+1} - \mu_k = \int_{-\infty}^{\infty} x d(-F^k(x)(1 - F(x))).$$

Integrating by parts the last expression we obtain

$$\mu_{k+1} - \mu_k = \int_{-\infty}^{\infty} F(x)^k (1 - F(x)) dx$$

with boundary terms vanishing thanks to the Lemma on page 37 of [3].

Since the variables are not Deltas, the last integral is positive and the result follows. \square

Remark 3.3. In the previous lemma it is necessary to assume the random variables to be independent. For example, if we consider a set of n applicants where j are valued 0 and $n - j$ are valued 1, then $\mu_k = 1$ for every $k \geq j + 1$.

Proposition 3.4. *The optimal threshold diverges with the number of candidates, that is, $c_*(n) \rightarrow \infty$ when $n \rightarrow \infty$.*

Proof. Let $c > 0$ be fixed. From (1.2), we have

$$\begin{aligned} \Delta V_n(c) &\geq \frac{\mu_c}{c(c+1)} + \mu_{c+1} \sum_{k=c+2}^{n-1} \frac{1}{k(k-1)} - \frac{\mu_c}{c} + \frac{\mu_1}{n-1} \\ &= \frac{\mu_{c+1} - \mu_c}{c+1} - \frac{\mu_{c+1} - \mu_1}{n-1}. \end{aligned}$$

Lemma 3.2 ensures that the first term above is strictly positive, while the last one tends to 0 as n tends to infinity, and so the derivative at c is strictly positive for n large enough. \square

4. EXAMPLES

In this section we work through different families of distributions. The problem of finding the optimal threshold is invariant under linear scalings of the applicants' qualities. For this reason the mean and variance of the random variables under consideration play no role in the estimates we provide.

In 4.1 we deal with some continuous distributions (Exponential, Normal and Pareto), and manage to prove that the upper bound from Theorem 2.1 is asymptotically optimal in a precise sense. In 4.2 we work out some discrete examples, recover the solution of the classical Secretary Problem and exhibit an example that shows that there is no non-trivial lower estimate for $c_*(n)$.

Plots of several of the examples considered, both for continuous and discrete distributions, are displayed in Figure 1. These illustrate the different behaviors $c_*(n)$ may exhibit.

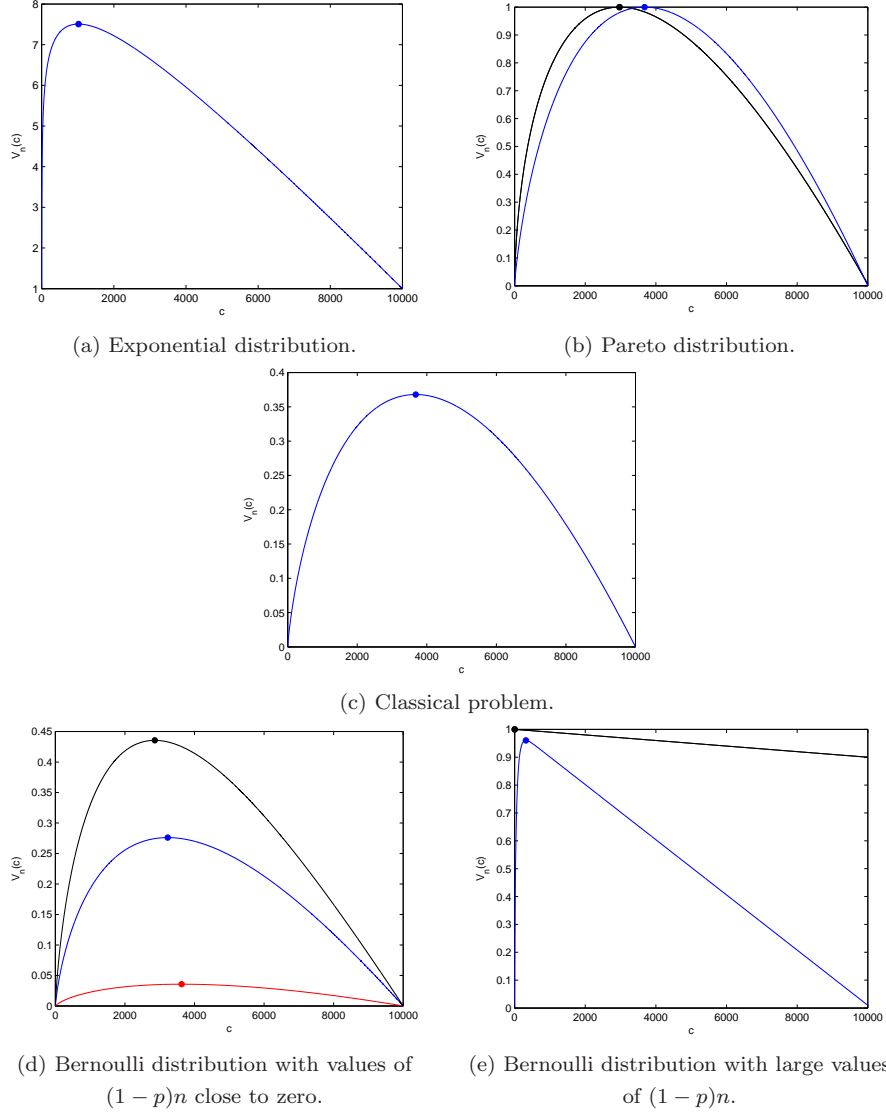


FIGURE 1. Plots of V_n as a function of c for different distributions with $n = 10000$. The maxima are highlighted. (a) In example 4.1, the maximum is attained at $c = 1022$, whereas $\frac{n}{\log(n)+\gamma} \sim 1021.7$. (b) Example 4.3, with $\alpha = 1.5$ (black) and $\alpha = 1 + 10^{-10}$ (blue), normalized in order to make $V_n(c_*) = 1$. The points where the maxima are attained approach to n/e as α tends to 1. (c) In Example 4.5, the optimal threshold is $c = 3680 \sim n/e$. Plots (d) and (e) are taken from Example 4.6. (d) The colors correspond to $p = 1 - 2/n$ (black), $p = 1 - 1/n$ (blue) and $p = 1 - 0.1/n$ (red). The points where the maxima are attained approach to n/e as p tends to 1. (e) Plots for $p = 0.1$ (black) and $p = 0.99$ (blue). The points where the maxima are attained tend to 2 as $p \rightarrow 0$.

4.1. Continuous i.i.d. random variables. Next we give three examples of i.i.d. continuous random variables that will suggest that ‘the heavier the weight of the tails, the longer the recruiter has to wait’. Let us recall that for uniform distributions it holds that $c_*(n) \sim \sqrt{n}$ (cf. [1]); here we provide other examples of interest.

Example 4.1 (Exponential distribution). Let the candidates’ values be given by independent exponential distributions, $X_k \sim \exp(1)$. It is easy to verify that

$$\mu_k = \sum_{j=1}^k \frac{1}{j},$$

thus,

$$\mu_{k+1} - \mu_k = \frac{1}{k+1}.$$

Therefore, formula (1.3) becomes

$$\begin{aligned} \Delta V_n(c) &= \sum_{k=c}^{n-1} \frac{1}{k(k+1)} - \frac{\sum_{k=1}^n \frac{1}{k} - 1}{n-1} \\ &= \frac{1}{c} - \frac{1}{n} + \frac{1}{n-1} - \frac{\sum_{i=1}^{n-1} \frac{1}{i}}{n-1}. \end{aligned}$$

From this last equation it is straightforward to check that

$$c_*(n) \sim \frac{n}{\log(n) + \gamma}$$

where

$$\gamma = \lim_n \sum_{k=1}^n \frac{1}{k} - \log n$$

is the Euler-Mascheroni constant. Indeed, setting $\Delta V_n(c) \leq 0$, we immediately bound

$$0 > \frac{1}{c} - \frac{\sum_{i=1}^{n-1} \frac{1}{i}}{n-1},$$

so that

$$c > \frac{n-1}{\sum_{i=1}^{n-1} \frac{1}{i}}$$

from which

$$c_*(n) > \left\lfloor \frac{n-1}{\sum_{i=1}^{n-1} \frac{1}{i}} \right\rfloor.$$

The bound

$$c_*(n) < \left\lceil \frac{n}{\sum_{i=1}^n \frac{1}{i}} \right\rceil$$

follows similarly.

Example 4.2 (Normal distribution). Assume the aspirants’ qualities are given by a distribution $N(0, 1)$. Then, the expected value of the maximum among the first k applicants satisfies

$$\mu_k^2 = \log \left(\frac{k^2}{2\pi} \right) - \log \log \left(\frac{k^2}{2\pi} \right) + f(k),$$

where f is a function such that $\lim_{k \rightarrow \infty} f(k) = 4\gamma$ (see [3, Ex. 10.5.3]). Therefore, if k is large enough,

$$(4.1) \quad \log\left(\frac{k^2}{2\pi}\right) - \log\log\left(\frac{k^2}{2\pi}\right) \leq \mu_k^2 \leq \log\left(\frac{k^2}{2\pi}\right).$$

This last inequality allows us to obtain an upper bound for the optimal threshold. Recall that, due to Proposition 3.4, $c_*(n) \rightarrow \infty$ as $n \rightarrow \infty$.

Resorting to (4.1) we write

$$\mu_c \Delta V_n(c) \leq \sum_{k=c+1}^{n-1} \frac{\mu_k^2}{k(k-1)} - \frac{\mu_c^2}{c} \leq \sum_{k=c+1}^{n-1} \frac{\log\left(\frac{k^2}{2\pi}\right)}{k(k-1)} - \frac{\log\left(\frac{c^2}{2\pi}\right)}{c} + \frac{\log\log\left(\frac{c^2}{2\pi}\right)}{c}.$$

Observe that

$$\sum_{k=c+1}^{n-1} \frac{\log\left(\frac{k^2}{2\pi}\right)}{k(k-1)} \leq \int_c^{n-1} \frac{\log\left(\frac{t^2}{2\pi}\right)}{(t-1)^2} dt,$$

whose right hand side equals to

$$\frac{\log\left(\frac{c^2}{2\pi}\right)}{c-1} + 2\log\left(\frac{c}{c-1}\right) - \frac{\log\left(\frac{(n-1)^2}{2\pi}\right)}{n-2} - 2\log\left(\frac{n-1}{n-2}\right).$$

Thus, since

$$\frac{\log\left(\frac{c^2}{2\pi}\right)}{c} < 1 \quad \text{and} \quad \log\left(\frac{c}{c-1}\right) < \frac{1}{c-1},$$

we bound

$$\begin{aligned} \mu_c \Delta V_n(c) &\leq 2\log\left(\frac{c}{c-1}\right) + \frac{\log\left(\frac{c^2}{2\pi}\right)}{c(c-1)} + \frac{\log\log\left(\frac{c^2}{2\pi}\right)}{c} - \frac{\log\left(\frac{(n-1)^2}{2\pi}\right)}{n-2} - 2\log\left(\frac{n-1}{n-2}\right) \\ &< \frac{3 + \log\log\left(\frac{c^2}{2\pi}\right)}{c-1} - \frac{\log\left(\frac{(n-1)^2}{2\pi}\right)}{n-2}. \end{aligned}$$

Finally, if we substitute c by

$$\tilde{c} = \frac{(n-2)\log\log n}{\log\left(\frac{(n-1)^2}{2\pi}\right)} + 1$$

in the expression above, we obtain

$$\Delta V_n(\tilde{c}) < \frac{\log\left(\frac{(n-1)^2}{2\pi}\right)}{\mu_{\tilde{c}}(n-2)} \left(\frac{3 + \log\log\left(\frac{\tilde{c}^2}{2\pi}\right)}{\log\log n} - 1 \right) \leq 0,$$

for n large enough. This provides an upper bound for the optimal threshold, namely

$$c_*(n) \leq \frac{(n-2)\log\log n}{\log\left(\frac{(n-1)^2}{2\pi}\right)} + 1.$$

Example 4.3 (Pareto distribution). Consider $(X_k)_{k=1,\dots,n}$ i.i.d. Pareto distributions with CDF equal to $F(x; \alpha) = 1 - x^{-\alpha}$ for $x \geq 1$, where $\alpha > 1$. In this case we have [3, pg. 52] that

$$\mu_k = \frac{\Gamma\left(1 - \frac{1}{\alpha}\right) \Gamma(k+1)}{\Gamma\left(k+1 - \frac{1}{\alpha}\right)}.$$

In [5] it is proved the estimates

$$(k+1)^{\frac{1}{\alpha}} \geq \frac{\Gamma(k+1)}{\Gamma(k+1-\frac{1}{\alpha})} \geq k^{\frac{1}{\alpha}}.$$

Recall that $\Gamma(x+1) = x\Gamma(x)$, and so

$$\mu_{k+1} - \mu_k = \frac{\mu_{k+1}}{\alpha(k+1)} \geq \frac{\Gamma(1-\frac{1}{\alpha})(k+1)^{\frac{1}{\alpha}-1}}{\alpha}.$$

Thus, taking into account (1.3), we obtain

$$\begin{aligned} \Delta V_n(c) &\geq \Gamma\left(1-\frac{1}{\alpha}\right) \left(-\frac{(n+1)^{\frac{1}{\alpha}}-1}{n-1} + \sum_{k=c}^{n-1} \frac{(k+1)^{\frac{1}{\alpha}-2}}{\alpha} \right) \\ &\geq \Gamma\left(1-\frac{1}{\alpha}\right) \left(-\frac{(n+1)^{\frac{1}{\alpha}}-1}{n-1} + \frac{1}{\alpha} \int_{c+1}^{n+1} x^{\frac{1}{\alpha}-2} dx \right). \end{aligned}$$

Note that for fixed α , when n is large enough, we have the inequality

$$-\frac{(n+1)^{\frac{1}{\alpha}}-1}{n-1} \geq -(n+1)^{\frac{1}{\alpha}-1},$$

therefore we can write

$$\Delta V_n(c) \geq \Gamma\left(1-\frac{1}{\alpha}\right) \left((c+1)^{\frac{1}{\alpha}-1} \frac{1}{\alpha-1} - (n+1)^{\frac{1}{\alpha}-1} \left(\frac{1}{\alpha-1} + 1 \right) \right).$$

Thus, we obtain that if

$$\frac{n+1}{c+1} \geq \alpha^{\frac{\alpha}{\alpha-1}}$$

then $\Delta V_n(c) \geq 0$. This implies that

$$\frac{n+1}{c_*(n)+1} \leq \alpha^{\frac{\alpha}{\alpha-1}}$$

whenever n large enough. Finally, note that this last term tends to e as α approaches 1, thus the bound of Theorem 2.1 is asymptotically attained for the Pareto distribution when $\alpha \rightarrow 1$.

4.2. Discrete random variables. In this paragraph we study the behavior of several discrete random variables. We also recover the solution of the classical Secretary Problem and provide a family of examples for which, as certain parameter varies, the threshold c_* exhibits both extremal behaviors, the lineal one limited by the upper bound from Theorem 2.1 and the constant one attained by the minimum possible of $c_*(n) = 2$ (see Remark 4.7 below).

Example 4.4 (Permutations). If the candidates' values $(X_k)_{1 \leq k \leq n}$ are distributed uniformly over the permutations of the numbers from 1 to n , these random variables are exchangeable and satisfy

$$\mu_k = \binom{n}{k}^{-1} \sum_{i=k}^n i \binom{i-1}{k-1}.$$

Since

$$\sum_{i=k}^n i \binom{i-1}{k-1} = k \binom{n+1}{k+1},$$

we obtain the simpler expression

$$\mu_k = \binom{n}{k}^{-1} k \binom{n+1}{k+1} = \frac{k(n+1)}{k+1},$$

and then conclude

$$\mu_{k+1} - \mu_k = \frac{n+1}{(k+2)(k+1)}.$$

This, together with (1.3), allows us to estimate the discrete derivative of V_n ,

$$\Delta V_n(c) = \sum_{k=c}^{n-1} \frac{n+1}{k(k+1)(k+2)} - \frac{n - \frac{n+1}{2}}{n-1}$$

and hence

$$\Delta V_n(c) = \frac{n^2 + n - c^2 - c}{2c(c+1)n} - \frac{1}{2} = \frac{n+1}{2c(c+1)} - \frac{1}{2n} - \frac{1}{2}$$

from which

$$c_*(n) = \left\lfloor \sqrt{n+1/4} + 1/2 \right\rfloor$$

is easily obtained.

It is worth noting that this kind of behavior is expected, as this situation is a discrete analogue of the uniform distribution.

Example 4.5 (Recovering the classical Secretary Problem). Consider random variables $(X_k)_{1 \leq k \leq n}$ in such a way that for each k , with probability $1/n$ it holds that X_k is equal to 1 and the rest of them are equal to 0. This is equivalent to the classical Secretary Problem since, in this case, maximizing the expected value corresponds to maximizing the probability of selecting the best applicant. Since $\mu_k = \frac{k}{n}$, applying equation (1.3) we obtain

$$\Delta V_n(c) = \frac{1}{n} \left(\sum_{k=c}^{n-1} \frac{1}{k} - 1 \right),$$

Therefore, $c_*(n)$ is the least integer c that satisfies $\sum_{k=c}^{n-1} \frac{1}{k} \leq 1$, as in the classical Secretary Problem.

Example 4.6 (Bernoulli variables). Assume the applicants' qualities are given by i.i.d. Bernoulli random variables $X_k \sim B(1, 1-p)$, so that $\mu_k = 1 - p^k$. Recalling formula (1.3), it follows that

$$\Delta V_n(c) = \sum_{k=c}^{n-1} \frac{p^k - p^{k+1}}{k} - \frac{p - p^n}{n-1} = (1-p) \sum_{k=c}^{n-1} \frac{p^k}{k} - \frac{p - p^n}{n-1}.$$

Since the function $k \mapsto \frac{p^k}{k}$ is decreasing, after a change of variables we obtain

$$\int_{c/n}^1 \frac{p^{nz}}{z} dz \leq \sum_{k=c}^{n-1} \frac{p^k}{k} \leq \int_{(c-1)/(n-1)}^1 \frac{p^{(n-1)z}}{z} dz.$$

We study the behavior of the optimal threshold in two different scenarios. For this purpose, let us consider $p = p(n)$ and define

$$f(n) := (1-p)n,$$

the expected number of candidates with quality $X_k = 1$. The two aforementioned situations are distinguished by the asymptotic behavior of $f(n)$ as $n \rightarrow \infty$.

In first place, assume that $\lim_n f(n) = \alpha \geq 0$ and perform calculations for $c = c(n)$. In such case, we have that $\lim_n p^n = e^{-\alpha}$ and

$$\lim_n \frac{p - p^n}{(n-1)(1-p)} = \begin{cases} \frac{1-e^{-\alpha}}{\alpha} & \text{if } \alpha > 0, \\ 1 & \text{if } \alpha = 0. \end{cases}$$

Let us call $g(\alpha)$ the function defined by the right hand side above. Then, given $\epsilon > 0$, if n is large enough we obtain

$$\frac{\Delta V_n(c)}{1-p} \geq \int_{c/n}^1 \frac{p^{nz}}{z} dz - g(\alpha) - \epsilon.$$

As the integral above is non convergent if the lower limit c/n is substituted by 0, if $c/n \rightarrow 0$ we have $\Delta V_n(c) \geq 0$. This shows that the optimal threshold is linear in n . A lower bound may be obtained if $c/n = \beta > 0$, since due to the Dominated Convergence Theorem,

$$\lim_n \int_{c/n}^1 \frac{p^{nz}}{z} dz = \int_{\beta}^1 \frac{e^{-\alpha z}}{z} dz.$$

Therefore,

$$\Delta V_n(n\beta) \geq \int_{\beta}^1 \frac{e^{-\alpha z}}{z} dz - g(\alpha) - 2\epsilon$$

if n is large enough. Taking β such that

$$\int_{\beta}^1 \frac{e^{-\alpha z}}{z} dz = g(\alpha),$$

we obtain a lower bound for c_* . For example, if $\alpha = 0$ we recover the optimal bound $\beta = 1/e$, and for $\alpha = 1$ we obtain $\beta \simeq 0.323$.

On the other hand, assuming that $\lim_n f(n) = \infty$ it is possible to show that the optimal threshold is not linear in n . Indeed, simple calculations give

$$(4.2) \quad \Delta V_n(c) \leq \frac{1}{n-c} \int_{(c-1)/(n-1)}^1 \left[(1-p)(n-c) \frac{p^{(n-1)z}}{z} - p + p^n \right] dz.$$

Assume that there exists an $\epsilon > 0$ such that $(c-1)/(n-1) > \epsilon$. Then, the integrand is easily shown to be negative for large n , because

$$(1-p)(n-c) \frac{p^{(n-1)z}}{z} - p + p^n \leq p \left[\frac{(1-p)(n-c)(n-1)p^{\epsilon(n-1)-1}}{c-1} - 1 + p^{n-1} \right] \leq 0$$

for large n , because the term in brackets tends to -1 as $n \rightarrow \infty$. This shows that if c is linear in n , the derivative at c is negative. Thus, the optimal threshold is not linear in n .

Remark 4.7. This last example provides a family of situations for which there is no lower bound for the optimal threshold. Indeed, let us fix n and $c > 2$ and consider the limit $p \rightarrow 0$. The integrand in the right hand side of (4.2) is pointwise bounded above by

$$p \left(\frac{(1-p)(n-c)(n-1)p^{c-2}}{c-1} - 1 + p^{n-1} \right),$$

and so it is negative for every $z \in [(c-1)/(n-1), 1]$ if p is small enough. This proves that $\Delta V_n(c) \leq 0$ for every $n, c > 2$ if p is small enough, and thus $c_* = 2$.

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